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D-brane on deformed $AdS_3 \times S^3$

Malak Khouchen and Josef Klusoň

Institute for Theoretical Physics and Astrophysics, Faculty of Science, Masaryk University, Kotlářská 2, 611 37, Brno, Czech Republic

E-mail: khouchen@mail.muni.cz, klu@physics.muni.cz

Abstract: We study D1-brane in $AdS_3 \times S^3$ κ-deformed background with non-trivial dilaton and Ramond-Ramond fields. We consider purely time-dependent and spatially-dependent ansatz where we study the solutions of the equations of motion for D1-brane in given background. We find that the behavior of these solutions crucially depends on the value of the parameter a that was introduced in [7].

Keywords: D-branes, AdS-CFT Correspondence

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Contents

1 Introduction

The recent developments in the field of higher-dimensional extended objects have led to the deep understanding of the superstrings and supergravity theories. D-branes have been by now well understood both from the conformal field theory (CFT) and from the geometric, target-space viewpoint.[1] Such hyperplanes are dynamical rather than rigid and they are defined by the property that open strings can end on them [3]. The incorporation of such D-branes permits to argue that the different types of string theories are different states of a single theory, which also contain states with arbitrary configurations of D-branes. The dynamics of Dp-brane is governed by the action

$$S = S_{\rm DBI} + S_{\rm WZ}\,, \tag{1.1}$$

where $S_{\rm DBI}$ is Dirac-Born-Infeld action

$$S = -T_p \int d^{p+1}\sigma e^{-\Phi}\sqrt{-\det(g_{\alpha\beta} + b_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta})}\,, \tag{1.2}$$

and $S_{\rm WZ}$ is Wess-Zumino action of the form

$$S_{\rm WZ} = T_p \int \sum_n C^{(n)} e^{(2\pi\alpha')F+b}\,, \tag{1.3}$$

[1] For review and extensive list of references, see for example [1, 2].

where $\sigma^\mu, \mu = 0, \dots, p$ label world-volume of Dp-brane, $\Phi(x)$ is the dilaton and $g_{\alpha\beta}, b_{\alpha\beta}$ given in (1.2) are the pull-backs of the target space metric and the NS-NS two form field to the world-volume of Dp-brane

$$g_{\alpha\beta} = g_{MN}\partial_\alpha x^M \partial_\beta x^N \, , \quad b_{\alpha\beta} = b_{MN}\partial_\alpha x^M \partial_\beta x^N \, , \tag{1.4}$$

where $x^M(\sigma)$ are embedding coordinates of $D1$-brane. Finally, $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is the field strength for the world-volume gauge field A_α. The coupling of Dp-brane to the Ramond-Ramond fields is expressed through the Wess-Zumino term (1.3) where it is understood that expressions given there are forms and the multiplications between them have the form of the wedge product.

Motivated by the recent surge of interest in finding out the dualities in D-branes and fundamental strings bound states in anti-de Sitter space and gaining more insight into CFT, we study in this paper the dynamics of $D1$-branes described by Dirac-Born-Infeld action and Wess-Zumino terms. The corresponding target-space geometries are three-dimensional κ-deformed $AdS_3 \times S^3$ space-time. Very interesting class of deformations of target space-time have been introduced in [14, 22] that preserve the integrability of the two-dimensional quantum field theory on the world sheet.[2] In the κ-deformed anti-de Sitter background model, the metric is a direct sum of the deformed AdS_n and S^n parts and could be truncated from the ten-dimensional metric to κ-deformed $AdS_3 \times S^3$ for example [4]. The presence of the deformation parameter κ introduces new interesting results that reproduce the ordinary undeformed case in the limit $\kappa \to 0$ as in [6]. Hence, it is interesting to study the dynamics of $D1$-brane in given background as well. In fact, recently a one-parameter model of the κ-deformed background $AdS_3 \times S^3$ with non-trivial Ramond-Ramond (RR) forms and dilaton was proposed in [7]. A remarkable property of given background is that it depends on parameter a where it is presumed that a is a particular function of κ while the full solutions were constructed for the special values $a = 0$ and $a = 1$ only. In our present work, we will use these one-parameter backgrounds to analyze static and time-dependent solutions of $D1$-brane equations of motion in the background with non-trivial dilaton and with RR fields. Our analysis will reveal subtle features. Specifically, for the κ deformed AdS_3 background with $a = 0$ we will show that $D1$-brane does not see the presence of the singularity of the κ-deformed background and can reach $\rho \to \infty$ limit. We also find that the static solutions are very simple deformations of the static solution known as AdS D1-brane in AdS_3 background with RR flux. Since such a solution has not been found in the global coordinates before we present this result in the appendix A. Moreover, appendix B exhibits static solutions of p$D1$-branes in AdS_3 space-time with non-trivial B_{NS} field. The case of $(1, q)$ string was analyzed previously in [18], but we provide an extended analysis here for the (p, q) string in order to see the S-duality between given solution and the static D1-brane solution in undeformed AdS_3 space-time with Ramond-Ramond fields.

We also consider static and time-dependent solutions of D1-brane equations of motion for the κ-deformed background when the value of the parameter a is equal to 1. In this case we find D1-brane cannot cross the singularity $\rho_c = \frac{L}{\kappa}$ and we also find that the static solutions is more complicated than in case $a = 0$.

[2]For further works, see [4–17].

The plan of this paper is as follows: in section 2 we will consider static $D1$-brane in the κ-deformed background $AdS_3 \times S^3$ [7]. We study the solutions of the corresponding equations of motions and discuss the possibility of the $D1$-brane to reach the $\rho \to \infty$ limit of the deformed $AdS_3 \times S^3$ space. In section 3 we consider a time-dependent ansatz. In conclusion 4 we present summary of our results and their possible extension. Finally, some details of the calculations are summarized in the appendices. In appendix A we find static D1-brane solutions in non-deformed $AdS_3 \times S^3$ background with non-trivial RR fields. In appendix B, we find static solutions of p$D1$-branes in non-deformed AdS_3 space-time with non-trivial B_{NS} field in global coordinates which is simple generalization of the solution found in [18].

2 D1-brane in κ-deformed background

In this section, we will study time-independent solutions of the equations of motion that follow from D-brane actions (1.2) and (1.3) in case when $D1$-brane is embedded in κ-deformed $AdS_3 \times S^3$ background [7] that has the form

$$\begin{aligned} ds^2 &= \frac{1}{1-\kappa^2\frac{\rho^2}{L^2}}\left[-\left(1+\frac{\rho^2}{L^2}\right)dt^2+\frac{d\rho^2}{1+\frac{\rho^2}{L^2}}\right]+\rho^2 d\chi^2 \\ &\quad +\frac{1}{1+\kappa^2\frac{r^2}{L^2}}\left[\left(1-\frac{r^2}{L^2}\right)d\varphi^2+\frac{dr^2}{1-\frac{r^2}{L^2}}\right]+r^2 d\psi^2\,, \end{aligned} \tag{2.1}$$

with non-trivial dilaton and Ramond-Ramond fields

$$\begin{aligned} a=0 \quad e^{-2\Phi} &= \frac{(1-\kappa^2\frac{\rho^2}{L^2})(1+\kappa^2\frac{r^2}{L^2})}{[1-(\frac{\kappa\rho r}{L^2})^2]^2}\,, \\ C^{(2)} &= \frac{1}{L}\frac{1}{1-(\kappa\frac{\rho r}{L^2})^2}\left[\rho^2(dt+\kappa L d\varphi)\wedge\left(d\chi+\kappa\frac{r^2}{L^2}d\psi\right)\right. \\ &\qquad\left.-r^2(Ld\varphi-\kappa dt)\wedge\left(d\psi+\kappa\frac{\rho^2}{L^2}d\chi\right)\right], \\ a=1: \quad e^{-2\Phi} &= \frac{(1-\kappa^2\frac{\rho^2}{L^2})(1+\kappa\frac{r^2}{L^2})}{[1+\frac{\kappa^2}{L^2}(r^2-\rho^2+r^2\rho^2)]^2}\,, \\ C^{(2)} &= \frac{\sqrt{1+\kappa^2}}{L(1+\frac{\kappa^2}{L^2}(r^2-\rho^2+\rho^2 r^2))}\left[\rho^2 dt\wedge d\chi+\kappa\left[r^2-\rho^2+\frac{1}{L^2}(\rho r)^2\right]dt\wedge d\varphi\right. \\ &\qquad\left.+\frac{\kappa}{L}(\rho r)^2 d\chi\wedge d\psi-r^2 L d\varphi\wedge d\psi\right], \end{aligned} \tag{2.2}$$

where L is the inverse curvature scale. Consider D1-brane in given background whose dynamics is governed by the action

$$S=-T_{D1}\int d^2\sigma e^{-\Phi}\sqrt{-\det g_{\alpha\beta}-(2\pi\alpha')^2F_{\tau\sigma}^2}+T_{D1}\int d^2\sigma C^{(2)}_{MN}\partial_\tau x^M\partial_\sigma x^N\,. \tag{2.3}$$

Generally, the equations of motion for x^M that follow from given action have the form

$$
\begin{aligned}
&-\partial_\alpha\left[T_{D1}e^{-\Phi}\frac{g_{MN}\partial_\beta x^N g^{\beta\alpha}\det g_{\alpha\beta}}{\sqrt{-\det g_{\alpha\beta}-(2\pi\alpha')^2F_{\tau\sigma}^2}}\right] \\
&\quad+T_{D1}e^{-\Phi}\partial_M\Phi\sqrt{-\det g_{\alpha\beta}-(2\pi\alpha')^2F_{\tau\sigma}^2}+\frac{T_{D1}}{2}e^{-\Phi}\frac{\partial_M g_{KL}\partial_\alpha x^K\partial_\beta x^L g^{\beta\alpha}}{\sqrt{-\det g_{\alpha\beta}-(2\pi\alpha')^2F_{\tau\sigma}^2}} \\
&\quad+T_{D1}\partial_M C^{(2)}_{KL}\partial_\tau x^K\partial_\sigma x^L-T_{D1}\partial_\alpha[\epsilon^{\alpha\beta}C^{(2)}_{MN}\partial_\beta x^N]=0\,,
\end{aligned}
\tag{2.4}
$$

where $\epsilon^{\tau\sigma}=-\epsilon^{\sigma\tau}=1$. On the other hand the equation of motion for A_α implies

$$
T_{D1}\frac{e^{-\Phi}(2\pi\alpha')^2F_{\tau\sigma}}{\sqrt{-\det g_{\alpha\beta}-(2\pi\alpha')^2F_{\tau\sigma}^2}}=\Pi\,,
\tag{2.5}
$$

where Π is constant that counts the number of fundamental strings. Using this result, we express $F_{\tau\sigma}$ as

$$
(2\pi\alpha')F_{\tau\sigma}=\frac{\Pi}{2\pi\alpha'}\frac{\sqrt{-\det g_{\alpha\beta}}}{\sqrt{T_{D1}^2e^{-2\Phi}+\frac{\Pi^2}{(2\pi\alpha')^2}}}\,.
\tag{2.6}
$$

Then the equations of motion (2.4) simplify as

$$
\begin{aligned}
&T_{D1}\partial_\alpha\left[\sqrt{e^{-2\Phi}+\frac{T_{F1}^2}{T_{D1}^2}\Pi^2}g_{MN}\partial_\beta x^N g^{\beta\alpha}\right]+T_{D1}^2e^{-2\Phi}\partial_M\Phi\frac{\sqrt{-\det g_{\alpha\beta}}}{\sqrt{T_{D1}^2e^{-2\Phi}+\frac{\Pi^2}{(2\pi\alpha')^2}}} \\
&\quad-\frac{T_{D1}}{2}\sqrt{e^{-2\Phi}+\frac{T_{F1}^2}{T_{D1}^2}\Pi^2}\partial_M g_{KL}\partial_\alpha x^K\partial_\beta x^L g^{\beta\alpha}\sqrt{-\det g_{\alpha\beta}} \\
&\quad+T_{D1}\partial_M C^{(2)}_{KL}\partial_\tau x^K\partial_\sigma x^L-T_{D1}\partial_\alpha[\epsilon^{\alpha\beta}C^{(2)}_{MN}\partial_\beta x^N]=0\,,
\end{aligned}
\tag{2.7}
$$

where $T_{F1}=\frac{1}{2\pi\alpha'}$. This is the form of the equation of motion for D1-brane that we will be our starting point. If we now return to the specific background given in (2.2) , we see that since the background depends on r through its square we find that the equation of motion for constant r has the form

$$
\frac{\delta\mathcal{L}}{\delta r^2}r=0
\tag{2.8}
$$

that has solution $r=0$. In the same way we can show that the equations of motions for φ and ψ have the solutions $\varphi=\psi=0$. In other words we will not consider solutions with non-trivial behavior on deformed $S^{(3)}$.

2.1 Static solutions

Let us now consider the static $D1$-brane solution when we assume the following ansatz

$$
x^0=\tau\,,\quad \chi=\sigma\,,\quad \rho=\rho(\chi)
\tag{2.9}
$$

so that

$$
g_{\tau\tau}=g_{tt}\,,\quad g_{\sigma\sigma}=g_{\chi\chi}+g_{\rho\rho}\rho'^2\,,\quad \rho'\equiv\frac{d\rho}{d\chi}\,.
\tag{2.10}
$$

For this ansatz the equation of motion for $x^0 = t$ is obeyed automatically. In order to solve for ρ it is more convenient to consider the equation of motion for χ since the background fields do not depend on χ explicitly. Then, we obtain

$$\sqrt{e^{-2\Phi} + \frac{T_{F1}^2}{T_{D1}^2}\Pi^2 g_{\chi\chi}}\frac{\sqrt{-g_{tt}}}{\sqrt{g_{\chi\chi} + g_{\rho\rho}\rho'^2}} + C^{(2)}_{\chi t} = C\,, \quad C = \text{constant}\,. \tag{2.11}$$

From given equation we derive the differential equation for ρ in the form

$$\rho'^2 = \frac{1}{g_{\rho\rho}}\left[-\frac{\left(e^{-2\Phi} + \frac{T_{F1}^2}{T_{D1}^2}\Pi^2\right)g_{tt}g_{\chi\chi}^2}{(C + C^{(2)}_{t\chi})^2} - g_{\chi\chi}\right]\,. \tag{2.12}$$

In the following, we will solve this equation for different background fields by considering two a-families.

2.1.1 The case $a = 0$

Let us begin with the case $a = 0$ so that we have the following background fields

$$e^{-\Phi} = \sqrt{1 - \frac{\kappa^2}{L^2}\rho^2}\,, \quad C^{(2)}_{t\chi} = \frac{1}{L}\rho^2\,. \tag{2.13}$$

Then (2.12) gives

$$\rho'^2 = \frac{\rho^4\left(1 + \frac{T_{F1}^2}{T_{D1}^2}\Pi^2 - \frac{\kappa^2}{L^2}\rho^2\right)\left(1 + \frac{\rho^2}{L^2}\right)^2}{C^2\left(1 + \frac{\rho^2}{CL}\right)^2} - \rho^2\left(1 - \frac{\kappa^2}{L^2}\rho^2\right)\left(1 + \frac{\rho^2}{L^2}\right)\,. \tag{2.14}$$

This equation can be integrated at least in principle. However, when we choose $C = L$ the given equation simplifies considerably

$$\frac{d\rho}{\rho\sqrt{\frac{\rho^2}{L^2}\left(\frac{T_{F1}^2}{T_{D1}^2}\Pi^2 + \kappa^2\right) - 1}} = d\chi \tag{2.15}$$

that has a solution

$$\rho^2 = \frac{L^2}{\left(\frac{T_{F1}^2}{T_{D1}^2}\Pi^2 + \kappa^2\right)}\frac{1}{\cos^2(\chi - \chi_0)}\,. \tag{2.16}$$

We choose the integration constant by requiring that D1-brane approaches $\rho \to \infty$ for $\chi \to 0$. Hence, the final result is

$$\rho^2 = \frac{L^2}{\left(\frac{T_{F1}^2}{T_{D1}^2}\Pi^2 + \kappa^2\right)}\frac{1}{\sin^2\chi}\,. \tag{2.17}$$

Surprisingly, we find that there is only mirror modification of the static solution of D1-brane in AdS_3 background with non-trivial RR fields that is presented in appendix (A), where this modification is given by the presence of the deformation parameter κ. Further, we also see that D1-brane can be stretched through the singularity $\rho_c = \frac{L}{\kappa}$ and can reach $\rho \to \infty$. This is very remarkable result especially in the light of the solution that we find in case $a = 1$.

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2.1.2 The case $a = 1$

In this case, the relevant components of the background fields at $r = 0$ are

$$e^{-2\Phi} = \frac{1}{1-\kappa^2\frac{\rho^2}{L^2}}, \quad C^{(2)}_{t\chi} = \frac{\sqrt{1+\kappa^2}}{1-\kappa^2\frac{\rho^2}{L^2}}\frac{\rho^2}{L} . \tag{2.18}$$

Again (2.12) gives

$$\rho'^2 = \frac{\left(1+\frac{\rho^2}{L^2}\right)^2\rho^4\left(\frac{1}{1-\kappa^2\frac{\rho^2}{L^2}}+\frac{T^2_{F1}}{T^2_{D1}}\Pi^2\right)}{\left(C+\frac{\sqrt{1+\kappa^2}}{1-\kappa^2\frac{\rho^2}{L^2}}\frac{\rho^2}{L}\right)^2} - \rho^2\left(1-\kappa^2\frac{\rho^2}{L^2}\right)\left(1+\frac{\rho^2}{L^2}\right) \tag{2.19}$$

We simplify this equation by choosing $C = \frac{L}{\sqrt{1+\kappa^2}}$. Hence, the previous equation has the form

$$\rho^2_{\text{max}}\rho_{\text{min}}\frac{d\rho}{\rho(\rho^2_{\text{max}}-\rho^2)\sqrt{\rho^2-\rho^2_{\text{min}}}} = d\chi\,, \tag{2.20}$$

where

$$\rho^2_{\text{max}} = \frac{L^2}{\kappa^2}\,, \quad \rho^2_{\text{min}} = \frac{L^2}{(1+\kappa^2)\Pi^2\frac{T^2_{F1}}{T^2_{D1}}} . \tag{2.21}$$

Solving this equation, we get

$$\tan^{-1}\frac{\sqrt{\rho^2-\rho^2_{\text{min}}}}{\rho_{\text{max}}} + \frac{\rho_{\text{min}}}{2\sqrt{\rho^2_{\text{max}}-\rho^2_{\text{min}}}}\ln\left[\frac{\sqrt{\rho^2_{\text{max}}-\rho^2_{\text{min}}}+\sqrt{\rho^2-\rho^2_{\text{min}}}}{\sqrt{\rho^2_{\text{max}}-\rho^2_{\text{min}}}-\sqrt{\rho^2-\rho^2_{\text{min}}}}\right] = \chi-\chi_0 . \tag{2.22}$$

We choose the integration constant χ_0 in such a way that for $\rho \to \rho_{\text{min}}$, $\chi \to \frac{\pi}{2}$. Then we obtain

$$\chi_0 = \frac{\pi}{2} . \tag{2.23}$$

From (2.22), we see that $D1$-brane does not reach the maximum value ρ_{max} for χ in the interval $\chi \in (0, 2\pi)$. More precisely, $D1$-brane reaches the maximum value at ρ_{max} for $\chi \to -\infty$ that implies that $D1$-brane has to wrap compact χ direction infinitely many times. We also see that now $D1$-brane does not cross the singularity at $\rho_c = \frac{L}{\kappa}$. In summary, we see qualitative different behaviors of these two solutions corresponding to the cases $a = 0$ and $a = 1$. We will also see this difference in case of pure time-dependent solutions that will be analyzed in the next section.

3 Time-dependent solution

In order to find time-dependent solution, we consider an ansatz

$$t = \tau\,, \chi = \sigma\,, \quad \rho = \rho(t) \tag{3.1}$$

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so that

$$g_{\tau\tau} = g_{tt} + g_{\rho\rho}(\dot{\rho})^2\,, \quad g_{\sigma\sigma} = g_{\chi\chi}\,. \tag{3.2}$$

For such ansatz, we find that the equation of motion for χ is automatically obeyed. On the other hand, the equation of motion for t gives

$$\sqrt{e^{-2\Phi} + \frac{T_{F1}^2}{T_{D1}^2}\Pi^2}\frac{g_{tt}\sqrt{g_{\chi\chi}}}{\sqrt{-g_{tt} - g_{\rho\rho}(\dot{\rho})^2}} - C^{(2)}_{t\chi} = C\,, C = \text{const} \tag{3.3}$$

that implies following differential equation for $\dot{\rho}$

$$\dot{\rho} = \frac{\sqrt{-g_{tt}}}{\sqrt{g_{\rho\rho}}}\sqrt{1 + \frac{g_{tt}g_{\chi\chi}\left(e^{-2\Phi} + \frac{T_{F1}^2}{T_{D1}^2}\Pi^2\right)}{(C + C^{(2)}_{t\chi})^2}}\,. \tag{3.4}$$

In the following we will consider two different one-parameter a-families of background fields.

3.1 The case $a = 0$

Substituting the fields of (2.13) in (3.4), we obtain

$$\dot{\rho} = \left(1 + \frac{\rho^2}{L^2}\right)\sqrt{1 - \frac{\rho^2\left(1 + \frac{\rho^2}{L^2}\right)\left(1 + \frac{T_{F1}^2}{T_{D1}^2}\Pi^2 - \kappa^2\frac{\rho^2}{L^2}\right)}{\left(1 - \kappa^2\frac{\rho^2}{L^2}\right)\left(C + \frac{\rho^2}{L}\right)^2}} \tag{3.5}$$

Let us impose the condition $C = L$ that simplifies the given equation considerably. For this condition, the turning point at which $\dot{\rho} = 0$ will be at

$$\rho_{\max} = \frac{L}{\sqrt{\Pi^2\frac{T_{F1}}{T_{D1}} + \kappa^2}} \tag{3.6}$$

which is less than $\frac{L}{\kappa}$. Hence, we see that in this case the $D1$-brane does not cross the singularity at $\rho_c = \frac{L}{\kappa}$.

On the other hand let us consider the case when $\Pi = 0$. From (3.5), we obtain

$$\dot{\rho} = \left(1 + \frac{\rho^2}{L^2}\right)\sqrt{1 - \frac{\rho^2\left(1 + \frac{\rho^2}{L^2}\right)}{\left(C + \frac{\rho^2}{L}\right)^2}} \tag{3.7}$$

It seems interesting that now the expressions containing the deformation parameter κ disappear. The turning point is at

$$\rho^2_{t.p.} = \frac{C^2}{1 - 2\frac{C}{L}} \tag{3.8}$$

this implies that in order to have real solution we have to demand that $C < \frac{L}{2}$. For $C = \frac{L}{2}$, we realize that the $D1$-brane reaches $\rho \to \infty$ asymptotically. More explicitly, in such case we can easily integrate the differential equation with the result

$$2\rho - L\tan^{-1}\frac{\rho}{L} = t\,, \tag{3.9}$$

and we see that for $t \to \infty$ $D1$-brane approaches ($\rho = \infty$).

Finally, for $C > \frac{L}{2}$ the $D1$-brane reaches $\rho \to \infty$ since the expression under the square root is then always positive without any restrictions on the radial coordinate ρ. In other words, $D1$-brane with zero electric field can probe the space-time beyond the singularity $\rho_c = \frac{L}{\kappa}$ as well.

3.2 The case $a = 1$

In this case, substituting the fields of (2.18) in (3.4), the differential equation has the form

$$\dot{\rho} = \left(1 + \frac{\rho^2}{L^2}\right)\sqrt{1 - \frac{(1+\frac{\rho^2}{L^2})\rho^2\Big(1 + \frac{T_{F1}^2}{T_{D1}^2}\Pi^2\Big(1 - \kappa^2\frac{\rho^2}{L^2}\Big)\Big)}{\Big(C\Big(1 - \kappa^2\frac{\rho^2}{L^2}\Big) + \sqrt{1+\kappa^2}\frac{\rho^2}{L}\Big)^2}} \tag{3.10}$$

We are interested in the special case when $\Pi = 0$. In this case, we find the turning point at

$$\rho_{t.p.}^2 = \frac{-(2CA-1) \pm \sqrt{(2CA-1)^2 - 4C^2(A^2 - \frac{1}{L^2})}}{2(A^2 - \frac{1}{L^2})} \tag{3.11}$$

where $A = \frac{\sqrt{1+\kappa^2}}{L} - C\frac{\kappa^2}{L^2}$. After the analysis of the expression under the square root (the discriminant), we realize that it is always positive. Then we have to consider two cases.

In the first case, $A^2 - \frac{1}{L^2} < 0$ then to have an overall positive quantity, we require a condition

$$C \in \left(L\Big(\frac{\sqrt{1+\kappa^2}-1}{\kappa^2}\Big), L\Big(\frac{\sqrt{1+\kappa^2}+1}{\kappa^2}\Big)\right) \tag{3.12}$$

In the second case we have $A^2 - \frac{1}{L^2} > 0$ that gives

$$C \in \left(-\infty, L\Big(\frac{\sqrt{1+\kappa^2}-1}{\kappa^2}\Big)\right) \cup \left(L\Big(\frac{\sqrt{1+\kappa^2}+1}{\kappa^2}\Big), \infty\right) \tag{3.13}$$

In this case however we have also to demand that

$$2CA - 1 < 0 \tag{3.14}$$

that implies

$$C \in \left[\left(-\infty, \frac{L}{2\kappa^2}(\sqrt{1+\kappa^2} - \sqrt{1-\kappa^2}\right) \cup \left(\frac{L}{2\kappa^2}(\sqrt{1+\kappa^2} + \sqrt{1-\kappa^2}), \infty\right)\right] \tag{3.15}$$

Now, however we find that the second condition is always obeyed since the second interval is included in the first. Therefore, there always exist real roots corresponding to the turning points $\rho_{r.t.}$.

Let us try to determine the value of the turning point for large $\frac{C}{L} \gg 1$ (large energy limit). In this case, we can write $A \approx -\frac{C}{L^2}\kappa^2$ and we obtain

$$\rho_{r.t.}^2 = \frac{L^2}{\kappa^2}\left(1 + \frac{L^2}{2C^2\kappa^4} \pm \frac{L}{C}\frac{\sqrt{1+\kappa^2}}{2\kappa^2}\right) \tag{3.16}$$

so that when we restrict to the terms linear in $\frac{L}{C} \ll 1$ we obtain two roots

$$\rho^2_{\mathrm{max}} = \frac{L^2}{\kappa^2}\left(1 + \frac{L}{C}\frac{\sqrt{1+\kappa^2}}{2\kappa^2}\right) ,$$
$$\rho^2_{\mathrm{min}} = \frac{L^2}{\kappa^2}\left(1 - \frac{L}{C}\frac{\sqrt{1+\kappa^2}}{2\kappa^2}\right) . \tag{3.17}$$

In other words we find two situations. In the first case D1-brane is in the region below the singularity $\rho^2_c = \frac{L^2}{\kappa^2}$ and can reach its turning point at ρ^2_{min} and then it returns back. In the second case, D1-brane is in the region $\rho^2 > \rho^2_{\mathrm{max}}$ i.e. beyond the singularity. However, it is important that in both of these cases, D1-brane cannot go through the singularity.

Finally, we compare this result with the analysis of the time-dependent solution of the fundamental string in κ-deformed background. Recall that the fundamental string is described by the Nambu-Gotto action

$$S = -T_{F1}\int d\tau d\sigma \sqrt{-\det g_{\alpha\beta}} , \qquad g_{\alpha\beta} = g_{MN}\partial_\alpha x^M \partial_\beta x^N . \tag{3.18}$$

The equation of motion for t for the time dependent ansatz again implies

$$g_{tt}g^{\tau\tau}\sqrt{-g_{\chi\chi}(g_{tt} + g_{\rho\rho}\dot{\rho}^2)} = C , \qquad C = \text{constant} . \tag{3.19}$$

Solving given equation for $\dot{\rho}$ we obtain

$$\dot{\rho} = \frac{1}{CL}\left(1 + \frac{\rho^2}{L^2}\right)\sqrt{\frac{(\rho^2_- - \rho^2)(\rho^2 + \rho^2_+)}{1 - \kappa^2\frac{\rho^2}{L^2}}} \tag{3.20}$$

where

$$\rho^2_- = \frac{L^2}{2}\left[-\left(1 + \kappa^2\frac{C^2}{L^2}\right) + \sqrt{\left(1 + \kappa^2\frac{C^2}{L^2}\right)^2 + 4\frac{C^2}{L^2}}\right] ,$$
$$\rho^2_+ = \frac{L^2}{2}\left[\left(1 + \kappa^2\frac{C^2}{L^2}\right) + \sqrt{\left(1 + \kappa^2\frac{C^2}{L^2}\right)^2 + 4\frac{C^2}{L^2}}\right] .$$

Note that for large C, ρ_- has the form

$$\rho^2_- = L^2\left(\frac{1}{\kappa^2} - \frac{L^2}{C^2\kappa^6}\right) \tag{3.21}$$

We see that the allowed regions for the propagation of the string is $(0, \rho^2_-)$ and $(\frac{L^2}{\kappa^2}, \infty)$. In other words, strings cannot cross the singularity at $\rho^2_c = \frac{L^2}{\kappa^2}$ when it is originally confined in the region around the point $\rho = 0$. On the other hand, for $\frac{C}{L} \ll 1$ we obtain $\rho^2_- \approx C^2 \ll L^2$ and the string is confined in the region around $\rho = 0$ there is no sign of the deformation of the target space-time.

4 Conclusion

In this paper we have studied the dynamics of $D1$-brane in κ-deformed $AdS_3 \times S^3$ background with non-trivial dilaton and Ramond-Ramond fields [7]. We have found that the background with $a = 0$ possesses many interesting properties. We have shown that the static solution of $D1$-brane in the presence of RR-charges can reach $\rho \to \infty$ limit of the deformed $AdS_3 \times S^3$ space-time and that given solution is a slight modification from the AdS $D1$-brane solution in undeformed $AdS_3 \times S^3$ background with Ramond-Ramond flux that is found in appendix (A). Moreover, it is also very interesting that the time dependent solution does not see the presence of the singularity at $\rho_c = \frac{L}{\kappa}$. In other words, $D1$-brane in deformed $AdS_3 \times S^3$ space-time can cross given singularity and reach $\rho \to \infty$. Hence, $D1$-brane can be considered as natural probe of given space-time.

These results are in sharp contrast with the case $a = 1$ where we have shown that the $D1$-brane does not reach the singularity. Explicitly, it was shown that in such conditions the $D1$-brane can move in the region beyond the singularity or in a region below the singularity, but it can not cross the singularity in both situations. The latter result was confirmed by analyzing the dynamics of the fundamental string in given background. Again, it was demonstrated that a string originally confined in the region around $\rho = 0$ can not cross the singularity.

We have further examined the static gauge ansatz of $pD1$-branes bound to q fundamental strings with non-trivial NS-NS flux in global coordinates. After solving the equations of motion, we were able to generalize Bachas result [18] for the constant C that determines the radius of AdS. We have shown that C is proportional to the number of fundamental strings in the bound state and inversely proportional to the number of $D1$-branes. Finally, we considered the time-dependent solution of $pD1$-branes bound to q fundamental strings in the same background. We were able to show that in the limit $T_{(p,q)} \to qT_F$ the fundamental string can indeed reach $\rho \to \infty$.

The present analysis can be extended in various directions. First of all it would be very interesting and challenging to study the dynamics of $pD1$-branes in the κ-deformed $AdS_3 \times S^3$ with complex deformation parameter. Further one can try to find the complete solution with arbitrary parameter a then proceed with a similar analysis as we did in this paper. It would be also interesting to perform analysis of D1-brane and fundamental string configurations that could describe Wilson loops in dual field theory. We hope to return to these problems in future.

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A D1-brane as probe of $AdS_3 \times S^3$ background with Ramond-Ramond background

In this appendix we consider the static solution of D1-brane equations of motions in the non deformed $AdS_3 \times S^3$ background with non-zero Ramond Ramond field. Let us be

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more explicit and consider the case of the near horizon limit of D1-D5 brane system that in global coordinates has the form[3]

$$\begin{aligned} ds^2 &= -\left(1+\frac{\rho^2}{L^2}\right)dt^2+\left(1+\frac{\rho^2}{L^2}\right)^{-1}d\rho^2+\rho^2d\varphi^2 \\ &\quad +L^2(d\theta^2+\cos^2\theta d\phi^2+\sin^2\theta d\chi^2)\,, \\ e^{\Phi} &= \mathcal{R}^2\,, \quad C^{(2)}_{t\varphi}=-\frac{Q_5}{L^3}\rho^2\,, \quad C^{(2)}_{\phi\chi}=Q_5\sin^2\theta\,, \end{aligned} \tag{A.1}$$

where $\varphi,\phi,\chi,y_m\in[0,2\pi]$ and $\theta\in[0,\pi]$ and where

$$\mathcal{R}^2=\sqrt{\frac{Q_1}{Q_5}}\,, \quad L^2=\sqrt{Q_1Q_5}\,. \tag{A.2}$$

The equations of motion for $D1$-brane in given background have the form

$$\begin{aligned} &\frac{T_{D1}}{2}\frac{e^{-\Phi}\partial_M g_{KL}\partial_\alpha x^K\partial_\beta x^L g^{\beta\alpha}\mathrm{det}g}{\sqrt{-\mathrm{det}g_{\alpha\beta}-(2\pi\alpha')^2F^2_{\tau\sigma}}}-\partial_\alpha\left[\frac{T_{D1}e^{-\Phi}g_{MN}\partial_\beta x^N g^{\beta\alpha}\mathrm{det}g}{\sqrt{-\mathrm{det}g-(2\pi\alpha')^2F^2_{\tau\sigma}}}\right] \\ &\qquad -\partial_\alpha[C_{MN}\partial_\beta x^N\epsilon^{\alpha\beta}]=0\,, \end{aligned} \tag{A.3}$$

while the equation of motion for A_α again implies

$$T_{D1}\frac{e^{-\Phi}(2\pi\alpha')^2F_{\tau\sigma}}{\sqrt{-\mathrm{det}g_{\alpha\beta}-(2\pi\alpha')^2(F_{\tau\sigma})^2}}=\Pi \tag{A.4}$$

Let us now presume an ansatz where the $D1$-brane is wrapping τ and φ directions and where $\rho=\rho(\sigma)$,

$$x^0=\tau\,,\varphi=\sigma\,,\rho=\rho(\varphi) \tag{A.5}$$

where we use the notation $\sigma^0=\tau,\sigma^1=\sigma$ keeping in mind that σ is dimensionless. For the ansatz (A.5), the equation of motion (A.3) for $M=0$ is automatically satisfied while that of φ implies

$$\sqrt{T^2_{D1}+\left(\frac{\Pi}{2\pi\alpha'}\right)^2}e^{-\Phi}g_{\varphi\varphi}\sqrt{-g}g^{\sigma\sigma}+T_{D1}\frac{Q_5}{L^3}\rho^2=C\,, \qquad C=\text{constant} \tag{A.6}$$

that can be solved for ρ' as

$$\rho'^2=\frac{\rho^4(1+\frac{\rho^2}{L^2})^2}{C'^2(1-\frac{K}{LC'}\rho^2)^2}-\rho^2\left(1+\frac{\rho^2}{L^2}\right)\,, \tag{A.7}$$

where $K^2=\frac{T^2_{D1}}{(T^2_{D1}+(\frac{\pi}{2\pi\alpha'})^2)}$ and where now $\rho'\equiv\frac{d\rho}{d\varphi}$. Let us now choose the constant C' in such a way that

$$C'=-KL \tag{A.8}$$

[3] We follow the conventions used in [19, 20].

when the equation above simplifies considerably

$$\rho' = \frac{\rho}{KL}\sqrt{\rho^2(1-K^2)-K^2L^2} \tag{A.9}$$

and hence we find the solution

$$\frac{\rho^2}{L^2} = \frac{K^2L^2}{1-K^2L^2}\frac{1}{\cos^2(\varphi-\varphi_0)} \ . \tag{A.10}$$

We again choose the integration constant that for $\varphi \to 0$, the system approaches $\rho \to \infty$ so that

$$\frac{\rho^2}{L^2} = \frac{1}{\Pi^2}\frac{T_{D1}^2}{T_{F1}^2}\frac{1}{\sin^2\varphi} \ . \tag{A.11}$$

This is the solution corresponding to AdS D1-brane in AdS_3 background with non-trivial RR fields. In the next appendix we show that given configuration is S-dual to the specific bound state of D1-branes and fundamental strings in AdS_3 background with non-trivial B_{NS} two form.

B pD1-branes in AdS_3 with B_{NS} field in global coordinates

Let us now consider a collection of pD1-branes in AdS_3 background with the metric

$$ds^2 = -(1+\frac{\rho^2}{L^2})dt^2 + \frac{1}{1+\frac{\rho^2}{L^2}}d\rho^2 + \rho^2 d\chi^2 \tag{B.1}$$

but with non-zero B_{NS} field in the form

$$B = \frac{\rho^2}{L}d\chi \wedge dt \ . \tag{B.2}$$

Since we are interested in the collective dynamics of the bound state of p D1-branes it is clear that given action is the standard DBI action multiplied with the number p so that

$$S = -pT_{D1}\int d\tau d\sigma\sqrt{-\det g_{\alpha\beta} - (b_{\tau\sigma}+2\pi\alpha' F_{\tau\sigma})^2} \ . \tag{B.3}$$

Note that the equations of motion for x^M that follow from given action have the form

$$\begin{aligned}
&-\partial_\alpha\left[pT_{D1}\frac{g_{MN}\partial_\beta x^N g^{\beta\alpha}\det g_{\alpha\beta}}{\sqrt{-\det g_{\alpha\beta}-(b_{\tau\sigma}+2\pi\alpha' F_{\tau\sigma})^2}}\right] + \frac{pT_{D1}}{2}\frac{\partial_M g_{KL}\partial_\alpha x^K\partial_\beta x^L g^{\beta\alpha}\det g_{\alpha\beta}}{\sqrt{-\det g_{\alpha\beta}-(b_{\tau\sigma}+2\pi\alpha' F_{\tau\sigma})^2}} \\
&-pT_{D1}\partial_\tau\left[\frac{(b_{\tau\sigma}+(2\pi\alpha')F_{\tau\sigma})b_{MN}\partial_\sigma x^N}{\sqrt{-\det g_{\alpha\beta}-(b_{\tau\sigma}+2\pi\alpha' F_{\tau\sigma})^2}}\right] + pT_{D1}\partial_\sigma\left[\frac{(b_{\tau\sigma}+(2\pi\alpha')F_{\tau\sigma})b_{MN}\partial_\tau x^N}{\sqrt{-\det g_{\alpha\beta}-(b_{\tau\sigma}+2\pi\alpha' F_{\tau\sigma})^2}}\right] \\
&\qquad +pT_{D1}\frac{(b_{\tau\sigma}+(2\pi\alpha')F_{\tau\sigma})\partial_M b_{KL}\partial_\tau x^K\partial_\sigma x^L}{\sqrt{-\det g_{\alpha\beta}-(b_{\tau\sigma}+2\pi\alpha' F_{\tau\sigma})^2}} = 0
\end{aligned} \tag{B.4}$$

The equation of motion for A_α implies

$$pT_{D1}\frac{b_{\tau\sigma}+2\pi\alpha' F_{\tau\sigma}}{\sqrt{-\det g_{\alpha\beta}-(b_{\tau\sigma}+2\pi\alpha' F_{\tau\sigma})^2}} = \frac{q}{2\pi\alpha'} \ , \tag{B.5}$$

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where q is the number of fundamental strings bound to p $D1$-branes. Note that using the previous result we can express $b_{\tau\sigma} + (2\pi\alpha')F_{\tau\sigma}$ as

$$b_{\tau\sigma} + (2\pi\alpha')F_{\tau\sigma} = \frac{qT_{F1}\sqrt{-\det g_{\alpha\beta}}}{\sqrt{p^2T_{D1}^2 + q^2T_{F1}^2}} \tag{B.6}$$

Let us now choose the following ansatz

$$t = \tau, \quad \chi = \sigma\,, \rho = \rho(\sigma)\ . \tag{B.7}$$

In this case we find that the equation of motion for t is automatically obeyed while the equation of motion for χ implies

$$\sqrt{(pT_{D1})^2 + \left(\frac{q}{2\pi\alpha'}\right)^2} g_{\chi\chi} g^{\sigma\sigma}\sqrt{-\det g} + \frac{q}{2\pi\alpha'} b_{\varphi t} = \sqrt{(pT_{D1})^2 + \left(\frac{q}{2\pi\alpha'}\right)^2} C\,, \tag{B.8}$$

where C is a constant. From given equation we obtain differential equation for ρ'

$$\rho'^2 = \frac{\rho^4\left(1 + \frac{\rho^2}{L^2}\right)^2}{\left(C - \frac{qT_{F1}}{\sqrt{p^2T_{D1}^2 + q^2T_{F1}^2}L}\rho^2\right)^2} - \rho^2\left(1 + \frac{\rho^2}{L^2}\right)\ . \tag{B.9}$$

If we choose the integration constant C to be equal to

$$C = -\frac{LqT_{F1}}{\sqrt{p^2T_{D1}^2 + q^2T_{F1}^2}} \tag{B.10}$$

we find simple differential equation for ρ

$$\rho' = \rho\sqrt{\frac{\rho^2}{L^2}\frac{p^2T_{D1}^2}{q^2T_{F1}^2} - 1} \tag{B.11}$$

that has solution

$$\frac{\rho}{L} = \frac{qT_{F1}}{pT_{D1}}\frac{1}{\sin\chi} \tag{B.12}$$

which is the generalization of the solution found in [18] to the case of the bound state of p D1-branes and q fundamental strings. Note that the special case when we have $p = \Pi$ D1-branes and $q = 1$ fundamental strings is S-dual to the solution found in the previous section which is the bound state of single D1-brane and Π fundamental strings. It is also instructive to consider time-dependent solution corresponding to the motion of the bound state of pD1-branes and q fundamental strings in given background when we consider an ansatz

$$x^0 = \tau\,, \quad \rho = \rho(\tau)\,, \quad \chi = \sigma\ . \tag{B.13}$$

Then the equation of motion for t implies

$$\sqrt{p^2T_{D1}^2 + q^2T_{F1}^2} g_{tt} g^{\tau\tau}\sqrt{-\det g_{\alpha\beta}} - qT_{F1}b_{t\chi} = C \tag{B.14}$$

and hence we obtain

$$\dot{\rho} = \frac{\sqrt{-g_{tt}}}{\sqrt{g_{\rho\rho}}}\sqrt{1 + \frac{(p^2T_{D1}^2 + q^2T_{F1}^2)g_{tt}g_{\chi\chi}}{(C - qT_{F1}b_{t\chi})^2}} \ . \tag{B.15}$$

For the background given in (B.1) and (B.2) we obtain that there is a turning point at

$$\rho^2 = \frac{-(p^2T_{D1}^2 + q^2T_{F1}^2 - 2\frac{C}{L}qT_{F1}) + \sqrt{(p^2T_{D1}^2 + q^2T_{F1}^2 - 2\frac{C}{L}qT_{F1})^2 + 4p^2\frac{C^2}{L^2}T_{D1}^2}}{2\frac{p^2T_{D1}^2}{L^2}} \ . \tag{B.16}$$

Note that there is a special formal case when $p = 0$ when the turning point occurs at

$$\rho^2 = \frac{C^2}{qT_{F1}(qT_{F1} - \frac{2C}{L})} \ . \tag{B.17}$$

We see that given turning point is real when $C < 2\frac{qT_{F1}}{L}$. We also see from (B.17) that the string can reach $\rho \to \infty$ when

$$C_{cr} = \frac{q}{2}T_{F1}L \ . \tag{B.18}$$

In fact, it is easy to see that for $C > C_{cr}$, the expression under the square root in (B.15) is always positive for all ρ. Hence, for $C > C_{cr}$ and for p the given configuration can always reach $\rho \to \infty$.

Finally, we would like to compare the given result with the analysis of the motion of fundamental strings in the given background. Recall that the dynamics of the classical string is governed by the Nambu-Gotto action

$$S_{NG} = -T_{F1}\int d\tau \int_0^l d\sigma \left[\sqrt{-\det g_{\alpha\beta}} + \frac{1}{2}\epsilon^{\alpha\beta}B_{MN}\partial_\alpha x^M\partial_\beta x^N\right] , \tag{B.19}$$

where $\epsilon^{\tau\sigma} = -\epsilon^{\sigma\tau} = 1$. Now the equation of motion of x^M takes the form

$$\partial_\alpha[g_{MN}\partial_\beta x^N g^{\beta\alpha}\sqrt{-\det g_{\alpha\beta}} + \epsilon^{\alpha\beta}B_{MN}\partial_\beta x^N] = 0 \ . \tag{B.20}$$

Consider the time-dependent ansatz as in case of $D1$-brane

$$x^0 = \tau \ , \quad \chi = \sigma \ , \quad g_{\sigma\sigma} = g_{\chi\chi} \ , \quad g_{\tau\tau} = g_{tt} + g_{\rho\rho}(\dot{\rho})^2 \ . \tag{B.21}$$

Then the equation of motion for χ is obeyed automatically while that for t implies

$$\dot{\rho}^2 = \left(1 + \frac{\rho^2}{L^2}\right)\left(1 - \frac{(1 + \frac{\rho^2}{L^2})\rho^2}{(C + \frac{\rho^2}{L^2})^2}\right) \tag{B.22}$$

Now the expression on the right has turning point at

$$\rho^2 = \frac{C^2}{1 - 2\frac{C}{L}} \Rightarrow C < \frac{L}{2} \tag{B.23}$$

Let us try to integrate the equation of motion for $C = \frac{L}{2}$ when we obtain

$$d\rho \frac{\left(1 + 2\frac{\rho^2}{L^2}\right)}{\sqrt{1 + \frac{\rho^2}{L^2}}} = dt \tag{B.24}$$

Integrating both sides we obtain

$$\frac{\rho^2}{L^2} = \frac{-1 + \sqrt{1 + (t - t_0)^2}}{2} \tag{B.25}$$

and we see that given string reaches $\rho = \infty$ in the limit $t \to \infty$. It is also easy to see that for $C > L/2$ there is no turning point and fundamental string always reaches $\rho \to \infty$ which is well known fact [18].

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